JOURNAL OF APPROXIMATION THEORY **30**, 309–314 (1980)

Error Estimates for Gauss–Jacobi Quadrature Formulae with Weights Having the Whole Real Line as Their Support

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Communicated by R. Bojanic

Received May 8, 1979

1. INTRODUCTION

Let $d\alpha$ be a non-negative measure having the whole real line or a part of the real line as its support. Let the support of $d\alpha$ contain infinitely many points and let $\int x^n \, d\alpha(x) < \infty$ for $n = 0, 1, 2, \ldots$.

Then there exists a uniquely determined sequence of orthonormal polynomials $\{P_n(d\alpha; x)\}$ with respect to this measure (see [1, Sect. I.1]); they are determined by the properties:

(a) $P_n(d\alpha; x) = \gamma_n(d\alpha)\, x^n + \cdots$ is a polynomial of degree n and $\gamma_n(d\alpha) = \gamma_n > 0$;

(b) $\int P_n(d\alpha)\, P_m(d\alpha)\, d\alpha = \delta_{mn}$, the Kronecker symbol.

It is well known that all zeros $x_{kn}(d\alpha) = x_{kn}$ $(k = 1, 2, \ldots, n)$ of $P_n(d\alpha; x)$ are real and are contained in the smallest interval overlapping the support of $d\alpha$.

The interpolatory quadrature formula

$$Q_n(d\alpha; f) \overset{\text{def}}{=} \sum_{k=1}^{n} \lambda_n(d\alpha; x_{kn})\, f(x_{kn}) \qquad \left(\sim \int f \, d\alpha\right) \tag{1.1}$$

has the property that, for every polynomial P_{2n-1} of degree $\leqslant 2n-1$,

$$Q_n(d\alpha; P_{2n-1}) = \int P_{2n-1}\, d\alpha.$$

The coefficients $\lambda_n(d\alpha; x_{kn})$ in this formula are called the Christoffel numbers and are given by

$$\lambda_n^{-1}(d\alpha; x) = \sum_{\nu=0}^{n-1} P_\nu^2(d\alpha; x).$$

0021-9045/80/120309-06$02.00/0

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Equation (1.1) is the Gauss–Jacobi quadrature formula. The nodes x_{kn} are called the Gaussian abscissae with respect to $d\alpha$.

If, in addition, $d\alpha$ is an absolutely continuous measure, then $d\alpha(x) = \alpha'(x)\,dx$ and $\alpha'(x)$ is a weight function. In this case, $\alpha'(x)$ will be denoted by $w(x)$ and $P_n(d\alpha)$ by $P_n(w)$.

Finally, x_{1n} will denote the greatest zero of $P_n(d\alpha)$.

2. Error Estimates for Entire Integrands

In order to prove our main results in this paper, we are going to use the following three lemmas due to G. Freud:

Lemma 2.1. *Let $f(z)$ be analytic in a domain D containing the Gaussian abscissae x_{kn} $(k = 1, 2,..., n)$ and $x_{j,n+1}$ $(j = 1, 2,..., n+1)$, then we have*

$$Q_{n+1}(d\alpha; f) - Q_n(d\alpha; f) = \frac{\gamma_{n+1}}{\gamma_n} \cdot \frac{1}{2\pi i} \oint_{C_n} \frac{f(z)}{P_n(d\alpha; z)\, P_{n+1}(d\alpha; z)}\, dz, \tag{2.1}$$

where $C_n \subset D$ is a simple closed curve containing the zeros of $P_{n+1}(d\alpha)$ in its interior. Consequently, the error term of the quadrature formula is expressible as

$$\int f\, d\alpha - Q_n(d\alpha; f) = \sum_{v=n}^{\infty} \frac{\gamma_{v+1}}{\gamma_v} \cdot \frac{1}{2\pi i} \oint_{C_v} \frac{f(z)}{P_v(d\alpha; z)\, P_{v+1}(d\alpha; z)}\, dz \tag{2.2}$$

Lemma 2.2. *Let $w_Q(x) = \exp\{-2Q(x)\}$, $-\infty < x < \infty$, be a weight function where $Q(x)$ is an even differentiable function, except possibly at $x = 0$, increasing for $x > 0$, for which $x^\rho Q'(x)$ is increasing for $\rho < 1$ then we have*

$$C_1 q_n \leqslant x_{1n} \leqslant C_2 q_n, \tag{2.3}$$

where C_1, C_2 do not depend on n and q_s $(s > 0)$ is the (unique) positive solution of the equtlon

$$q_s Q'(q_s) = s. \tag{2.4}$$

LEMMA 2.3. *For every even weight function $w(x)$ we have*

$$\max_{1 \leqslant k \leqslant n-1} \frac{\gamma_{k-1}}{\gamma_k} \leqslant x_{1n} \leqslant 2 \max_{1 \leqslant k \leqslant n-1} \frac{\gamma_{k-1}}{\gamma_k}. \tag{2.5}$$

For the proofs of Lemmas 2.1, 2.2 *and* 2.3 *see* [2, 4 *and* 3], *respectively.*

Let W be the class of all weight functions of the form $w_Q(x) = \exp\{-2Q(x)\}$, $-\infty < x < \infty$, *where*

(i) *$Q(x)$ is an even, differentiable function, except possibly at $x = 0$, increasing for $x > 0$.*

(ii) *There exists $\rho < 1$ such that $x^\rho Q'(x)$ is increasing and*

(iii) *the sequence $\{q_n\}$ determined by* (2.4) *satisfies the condition*

$$\frac{q_{2n}}{q_n} \geqslant C_3 > 1, \qquad (n = 1, 2, \ldots) \tag{2.6}$$

for some constant C_3 independent of n.

Remarks.

(a) Observe that whenever $Q(x) = Q_\alpha(x) = \frac{1}{2}|x|^\alpha$ $(\alpha \geqslant 1)$, then $w_Q \in W$.

(b) We also remark, see [5], that (2.6) will be satisfied if we assume that $Q''(x)$ exists and $xQ''(x)/Q'(x) \leqslant C$, $-\infty < x < \infty$, $C = \text{constant}$.

We now turn to our main results.

THEOREM 2.1. *Let $w_Q \in W$. Then, there exists a constant $A \in (0, 1)$, depending on Q only, such that whenever $f(z)$ is an entire function satisfying*

$$\overline{\lim_{R \to \infty}} \frac{\max_{|z|=R}(\log |f(z)|)}{2Q(R)} \leqslant A \tag{2.7}$$

we have

$$\overline{\lim_{n \to \infty}} \left(\left| \int_{-\infty}^{\infty} f(x)\, e^{-2Q(x)}\, dx - Q_n(w_Q; f) \right| \right)^{1/n} < 1. \tag{2.8}$$

THEOREM 2.2. *Let $w_Q \in W$. Let $f(z)$ be an entire function satisfying*

$$\lim_{R \to \infty} \frac{\max_{|z|=R}(\log |f(z)|)}{2Q(R)} = 0. \tag{2.7$'$}$$

Then

$$\lim_{n\to\infty}\left(\left|\int_{-\infty}^{\infty} f(x)\,e^{-2Q(x)}\,dx - Q_n(w_Q;f)\right|\right)^{1/n} = 0. \qquad (2.8)'$$

3. Proof of Theorems 2.1 and 2.2

Proof of Theorem 2.1. The proof of this theorem will be mainly based on estimating the error form in (2.2).

From the assumption (ii), we can easily see that there exists an absolute constant C_4 such that

$$\frac{Q(x)}{xQ'(x)} \leqslant C_4, \qquad \text{for all} \quad x \geqslant x_0 > 0. \qquad (3.1)$$

Suppose that (2.7) holds with $A = 2^{-(M+1)}$, where M is equal to (say) the greatest integer not exceeding $1 + (1 + 2C_4 + \log C_2)/\log C_3$, with C_2, C_3 and C_4 as in (2.3), (2.6) and (3.1), respectively.

Let us also choose R_n such that

$$R_n Q'(R_n) = 2^M(n+1) \qquad (n = 1, 2, \dots). \qquad (3.2)$$

From (2.4), (3.2) and (2.6) we conclude

$$R_n \geqslant C_3^M q_{n+1} \qquad (n = 1, 2, \dots). \qquad (3.3)$$

Since w_Q is an even weight function, it follows that (see [6, Sect. 2.3(2)]):

$$\begin{aligned} P_n(w_Q; z) &= \gamma_n z^{n-2[n/2]} \prod_{k=1}^{[n/2]} (z^2 - x_{kn}^2) \\ &= \gamma_n z^n \exp\left\{\sum_{k=1}^{[n/2]} \log\left(1 - \frac{x_{kn}^2}{z^2}\right)\right\}, \end{aligned}$$

where z is any complex number and $x_{1n} > x_{2n} > \cdots > x_{[n/2],n}$ are the positive zeros of $P_n(w_Q)$. Thus

$$\begin{aligned} \frac{1}{|P_n(w_Q; z)|} &\leqslant \frac{1}{\gamma_n |z|^n} \exp\left\{-\sum_{k=1}^{[n/2]} \log\left(1 - \frac{x_{kn}^2}{|z|^2}\right)\right\} \\ &\leqslant \frac{1}{\gamma_n |z|^n} \exp\left\{\sum_{k=1}^{[n/2]} \frac{x_{kn}^2/|z|^2}{1 - (x_{kn}^2/|z|^2)}\right\} \end{aligned} \qquad (3.4)$$

whenever $x_{kn}^2/|z|^2 < 1$ $(k = 1, 2, \dots, [n/2])$.

By combining (2.1), (2.7) and (3.4), denoting by I_n the expression

$$\frac{\gamma_{n+1}}{\gamma_n}\frac{1}{2\pi i}\oint_{C_n}\frac{f(z)}{P_n(w_Q:z)\,P_{n+1}(w_Q;z)}\,dz,$$

and taking the path of integration to be the circle of radius R_n and center at the origin, we obtain that there exists a positive number N such that

$$|I_n| \leqslant \frac{1}{\gamma_n^2}\cdot\frac{1}{R_n^{2n}}\cdot\exp\left\{2\cdot 2^{-M}Q(R_n)+(n+1)\frac{(x_{1,n+1}^2/R_n^2)}{1-(x_{1,n+1}^2/R_n^2)}\right\},$$

for all $n \geqslant N$. By using (2.5), (2.3), (3.1), (3.2), (3.3) and the choice of M, we conclude

$$\begin{aligned}|I_n| &\leqslant \frac{1}{\gamma_0^2}\cdot C_2^{2n}\cdot\frac{\prod_{k=2}^{n+1}q_k^2}{C_3^{2Mn}q_{n+1}^{2n}}\exp\left\{2\frac{Q(R_n)}{R_nQ'(R_n)}(n+1)+(n+1)\right\}\\ &\leqslant \frac{1}{\gamma_0^2}\left\{\frac{C_2^2\exp(4C_4+2)}{C_3^{2M}}\right\}^n \qquad (n \geqslant N).\end{aligned}$$

And

$$B=\frac{C_2^2\exp(4C_4+2)}{C_3^{2M}}<1.$$

Hence

$$|I_n| \leqslant \frac{1}{\gamma_0^2}\cdot B^n \qquad (n \geqslant N). \tag{3.5}$$

By using (2.2), (3.5) and denoting $\int_{-\infty}^{\infty} f(x)\,e^{-2Q(x)}\,dx - Q_n(w_Q;f)$ by Δ_n, we obtain $|\Delta_n| \leqslant 1/\gamma_0^2\cdot\sum_{\nu=n}^{\infty}B^\nu \leqslant KB^n$; $n \geqslant N$ and K is a constant independent of n.

Therefore

$$\overline{\lim_{n\to\infty}}\,|\Delta_n|^{1/n} \leqslant \overline{\lim_{n\to\infty}}\,(B\,K^{1/n})=B<1.$$

This completes the proof of Theorem 2.1.

Proof of Theorem 2.2. From the above proof of Theorem 2.4 and condition (2.7)′, we can, by choosing M sufficiently large, state that:

For any $\varepsilon > 0$, there exists a positive number N, such that

$$|\Delta_n|^{1/n} \leqslant K\varepsilon, \qquad \text{for all} \quad n \geqslant N.$$

From this last inequality we can immediately conclude (2.8)′, which completes the proof of Theorem 2.2.

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